Storgruppsövning 21/11-13

5.89

$$P(\underbrace{X_{n+1}=k_1,\dots,X_{n+m}=k_m}_{(1)} \mid \underbrace{X_0=i_0,\dots,X_n=i}_{(2)}) \overset{?}{=} P(X_1=k_1,\dots,X_m=k_m \mid X_0=i)$$

Solution:

$$P((1)\mid(2)) = \frac{P((1),(2))}{P((2))} = \underbrace{\frac{P((1),(2))}{P(\dots(1),(2))}}_{P(X_{n+m}=k_m \mid X_{n+m-1}=k_{m-1})} \cdot \underbrace{\frac{P(\dots(1),(2))}{P(\dots(1),(2))}}_{(*)} \cdot \underbrace{\frac{P(\dots(1),(2))}{P(\dots(1),(2))}}_{(+)} \cdot$$

$$\cdots \underbrace{\frac{P(\dots(1),(2))}{P((2))}}_{P(X_{n+1}=k_1 \mid X_n=i)} = P(X_m=k_m, X_{m-1}=k_{m-1},\dots,X_1=k_1 \mid X_0=i) \qquad \blacksquare$$

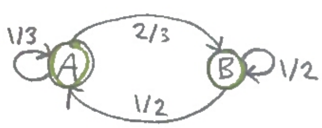
$(*) = P(X_{n+m-1}=k_{m-1} \mid X_{n+m-2}=k_{m-2})$

$(+) = P(X_{n+m-2}=k_{m-2} \mid X_{n+m-3}=k_{m-3})$

5.92

Producers A and B control market.
Present (time=0) A has 60%, B has 40% of market.
A looses 2/3 of market each year to B.
B —"— 1/2 ——"—— A.
What proportions of market do A and B hold after 2 years?

Solution:



$$P = \begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix}$$

$$\mu^{(0)} = \begin{pmatrix} 0.6 & 0.4 \end{pmatrix}$$
(A, B)

$$\mu^{(2)} = \mu^{(0)} P^2 = (0.6 \quad 0.4)\begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1/3 & 2/3 \\ 1/2 & 1/2 \end{pmatrix} = (0.6 \quad 0.4)\begin{pmatrix} 4/9 & 5/9 \\ 5/12 & 7/12 \end{pmatrix} =$$

$$= \dots = (0.43 \quad 0.57)$$

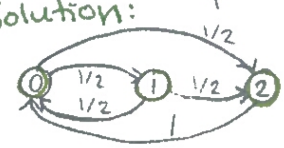
5.93

$S = \{0,1,2\}$

$$P = \begin{pmatrix} 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \\ 1 & 0 & 0 \end{pmatrix}$$

Is state 0 periodic?

Solution:



$d(0) = GCD\{n \geq 1 : P_{00}(n) > 0\} =$

$= GCD\{2,3,\dots\} = 1$

$\Longrightarrow$ not periodic.

5.95

Find stationary distribution for $P = \begin{pmatrix} 0.6 & 0.2 & 0.2 \\ 0.4 & 0.5 & 0.1 \\ 0.6 & 0 & 0.4 \end{pmatrix}$

Solution:

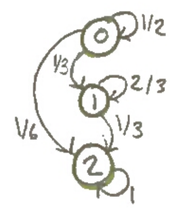
$\begin{cases} \pi P = \pi \\ \pi \text{ probability PMF} \end{cases}$

$$\begin{cases} 0.6\pi_0 + 0.4\pi_1 + 0.6\pi_2 = \pi_0 \Rightarrow \pi_2 = 0.4\pi_0 \\ 0.2\pi_0 + 0.5\pi_1 = \pi_1 \Rightarrow \pi_1 = 0.4\pi_0 \\ 0.2\pi_0 + 0.1\pi_1 + 0.4\pi_2 = \pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \Rightarrow \pi_0 = \frac{1}{1.8} \\ \pi_i \geq 0 \end{cases}$$

$$\Longrightarrow (\pi_0 \;\; \pi_1 \;\; \pi_2) = \left(\frac{5}{9} \;\; \frac{2}{9} \;\; \frac{2}{9}\right)$$

Comp. problem

$S = \{0, 1, 2\}$, $\mu^{(0)} = (1 \;\; 0 \;\; 0)$ $\quad P = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 2/3 & 1/3 \\ 0 & 0 & 1 \end{pmatrix}$



Find approximately $E(T)$ for $T = \min\{n \geq 1 : X_n = 2\}$.

Solution:

Reps $= 10^{10}$; Result $= 0$

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For (i=1:Reps)[
  x=0
  while(x=0)[
      Result=Result+1; Slump=Rand(0,1)
      If (Slump<1/2) x=0
      If (1/2≤Slump<5/6) x=1
      If (5/6≤Slump) x=2]
  While (x=1)[
      Result=Result+1; Slump=Rand(0,1)
      If(Slump<2/3) x=1
      If(2/3≤Slump) x=2]]

Write (Result/Reps)
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6.1.1

Let X_n be independent discrete r.v. Show X_n Markov chain! When is chain homogenous?

Solution:

$$P(\underbrace{X_{n+1} = j}_{1} \mid \underbrace{X_n = i, X_{n-1} \dots}_{2}) = \frac{P(\boxed{\;1\;}, \boxed{\;2\;})}{P(\boxed{\;2\;})} =$$

$$= \frac{P(\boxed{\;})P(\boxed{\;})\dots P(\boxed{\;})}{\underbrace{P(\boxed{\;})\dots P(\boxed{\;})}_{2}} = P(X_{n+1} = j) \Longrightarrow \text{Markov!}$$

Homogenous means no n-dependence $\Leftrightarrow$ X_n identically dist.

6.1.2

Dice rulled repeatedly.

a) X_n largest number shown up to ninth roll

b) N_n total number of sixes in n rolls.

c) C_n is the time since must reacent six at time n.

d) B_n is —ıı— until next six at time n.

Solution:

a) $P(X_{n+1}=j \mid X_n=i, \dots) = = \begin{cases} 1/6 & j>i>6 \\ i/6 & j=i<6 \\ 0 & j<i<6 \end{cases}$

$= \begin{cases} 1 & j=i=6 \\ 0 & j<i=6 \end{cases}$

b) $P(N_{n+1}=j \mid N_n=i, \dots) = \begin{cases} 1/6 & j=i+1 \\ 5/6 & j=i \\ 0 & \text{otherwise} \end{cases}$

c) $P(C_{n+1}=j \mid C_n=i, \dots) = \begin{cases} 1/6 & j=0 \\ 5/6 & j=i+1 \\ 0 & \text{otherwise} \end{cases}$

d) $B_{n+1} = \begin{cases} B_n - 1 \text{ if } B_n>0 \\ \text{geometically dist. waiting time dist.} & \text{if } B_n=0 \end{cases}$

$\Rightarrow P(B_{n+1}=j \mid B_n=i, \dots) = \begin{cases} 1 \text{ if } j=i-1 \\ (5/6)^{j-1}\, 1/6 \text{ if } i=0,\ j\geq 1 \end{cases}$

6.1.4 a)

X_n Markov Chain, $n_1, n_2 \dots$ increasing seqv. of integers ≥ 0

$Y_r = X_{n_r}$. Show Y_r Markov!

Solution:

look at eq. 6.1.2 in G&S.

$P(X_{n_{r+1}}=k_{r+1} \mid X_{n_r}=k_r, \dots X_{n_1}=k_1) \overset{?}{=} P(X_{n_{r+1}}=k_{r+1} \mid X_{n_r}=k_r)$

Find P^Y if $n_r=2r$ and $X_n = \sum_{i=1}^{n} Y_i$ where $Y_i = \begin{cases} 1 & \text{wp } p \\ -1 & \text{wp } q=1-p \end{cases}$

$P_{ij}^{Y} = \begin{cases} 2 & \text{wp } p^2 \\ 0 & \text{wp } 1-p^2-q^2=2qp \\ -2 & \text{wp } q^2 \end{cases}$

6.1.10

$P(\underbrace{X_r=k}_{\boxed{2 \text{ now}}} \mid \underbrace{X_i=x_i \text{ for } i=1,2,\dots,r-1}_{\boxed{1 \text{ history}}}, \underbrace{r+1,\dots,n}_{\boxed{3 \text{ future}}}) = P(X_r=k \mid X_{r-1}=x_{r-1}, X_{r+1}=x_{r+1})$

Solution:

$\dfrac{P(\boxed{3}\ \boxed{2}\ \boxed{1})}{P(\boxed{3}\ \boxed{1})} = \dfrac{P(\boxed{3}\ \boxed{2}\ \boxed{1})}{\sum_x P(\boxed{3}\ X_r=x\ \boxed{1})} =$

$$= \frac{\mu^{(1)}_{x_1} P_{x_1 x_2} \cdots P_{x_{r-1}k} P_{k x_{r+1}} \cdots P_{x_{n-1}x_n}}{\sum_x \mu^{(1)}_{x_1} P_{x_1 x_2} \cdots P_{x_{r-1}x} P_{x x_{r+1}} \cdots P_{x_{n-1}x_n}} = \frac{\mu^{(r-1)}_{x_{r-1}} P_{x_{r-1}k} P_{x_{r+1}k}}{\sum_x \mu^{(r-1)}_{x_{r-1}} P_{x_{r-1}x} P_{x x_{r+1}}} =$$

$$= \frac{P(X_{r+1}=x_{r+1}, X_r=k, X_{r-1}=x_{r-1})}{\sum_x P(X_{r+1}=x_{r+1}, X_r=x, X_{r-1}=x_{r-1})} = \frac{P(X_{r+1}=x_{r+1}, X_r=k, X_{r-1}=x_{r-1})}{P(X_{r+1}=x_{r+1}, X_{r-1}=x_{r-1})} =$$

$$= P(X_r=k \mid X_{r+1}=x_{r+1}, X_{r-1}=x_{r-1}) \quad ▨$$

6.1.12

P transition prob. matrix $\Longrightarrow$ P stocastic $\sum_j P_{ij} = 1 \quad \forall i$

P double stoc. if $\sum_i P_{ij} = 1 \quad \forall j$

P substoc. if $\sum_i P_{ij} \leq 1 \quad \forall j$

P stoc./dble stoc./sub.stoc. $\Longrightarrow P^n \ldots/\ldots/\ldots$

Solution:

$$\sum_j (P^{n+1})_{ij} = \sum_j \sum_k P^n_{ik} P_{kj} = \sum_k \underbrace{\sum_j P^n_{ik} P_{kj}}_{=1} = 1 \quad , \; P^n \text{ stoc.} \Longrightarrow P^{n+1} \text{ stoc.}$$

$$\sum_i (P^{n+1})_{ij} = \sum_i \sum_k P^n_{ik} P_{kj} = \sum_k \sum_i P^n_{ik} P_{kj} \quad , \; P^n \text{ dble stoc./sub} \Longrightarrow P^{n+1} \text{ dble stoc./sub}$$

6.2.1

Same as proof of thm 6.2.3 except trivial modifications.

6.2.2

X_n Markov chain with $\underbrace{\text{absorbing state } s}_{P_{ss}=1}$ with which all other states i communicate, that is $P_{is}(n) > 0$ for some $n = n(i)$ $\forall i$

Show that all states other than s are transient.

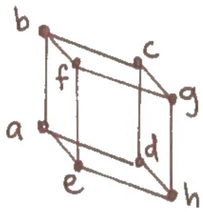
Solution:

For each i let $n_i = \min\{n : p_{is}(n) > 0\}$

Then if $X_0 = i$ and $X_{n_i} = j$ then X_n do not visit i for n in $\{1, \ldots, n_{i-1}\}$

Therefor this is an event wp>0 that leads to escape from i.

6.3.4



$$\begin{cases} \mu_a = 1 + 3/4\, \mu_{ba} \\ \mu_{ba} = 1 + 1/4\, \mu_{ba} + 1/2\, \mu_{ca} \\ \mu_{ca} = 1 + 1/4\, \mu_{ca} + 1/2\, \mu_{ba} + 1/4\, \mu_{ga} \\ \mu_{ga} = 1 + 1/4\, \mu_{ga} + 3/4\, \mu_{ca} \end{cases}$$

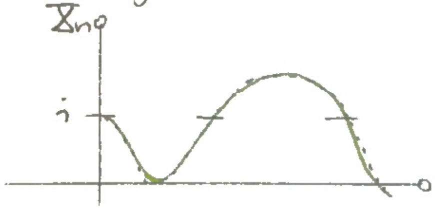
the solution comes later... Solve to get what is asked for!

6.2.3

Show that a state i is persistent iff the mean number of visits of the chain to i having started at i is infinite.

Solution:

Take your outcome ω of your random experiment



6.3.2

Random walk

$$\begin{cases} P_{i,i+2} = p \\ P_{i,i-1} = q = 1-p \end{cases}$$

When is $\underbrace{X_n}_{\text{every state } i \text{ of}}$ persistent?

Solution:

i persistent iff $\sum_{n=1}^{\infty} P_{ii}(n) = \infty$

$$P_{ii}(n) = \begin{cases} \binom{3k}{k} p^k (1-p)^{2k} & , n = 3k \\ 0 & \text{otherwise} \end{cases}$$

is $\sum_{k=1}^{\infty} \binom{3k}{k} p^k (1-p)^{2k} \begin{matrix} = \infty \\ < \infty \end{matrix}$?

$$\binom{3k}{k} p^k (1-p)^{2k} = \frac{(3k)!\, p^k (1-p)^{2k}}{k!\,(2k)!} \approx \{\text{Sterling}\} \approx \frac{(3k)^{3k+1/2}\, \cancel{e^{-3k}}\, \cancel{\sqrt{2\pi}}\, \sqrt{2\pi}\, p^k (1-p)^{2k}}{\cancel{\sqrt{2\pi}}\, k^{k+1/2} (2k)^{2k+1/2}\, \cancel{e^{-k}}\, \cancel{e^{-2k}}}$$

$$= \frac{\sqrt{2\pi}\sqrt{3/2}\left(\frac{27}{4} p(1-p)^2\right)^k}{\sqrt{k}}$$

$$\frac{d}{dp} \frac{27}{4} p(1-p)^2 = \frac{27}{4}(1-p)^2 - \frac{54}{4} p(\cancel{1-p}) = 0 \quad \text{max for } p = 1/3$$

$$\begin{cases} \frac{27}{4} p(1-p)^2 = 1 & p = 1/3 \\ \text{— '' —} < 1 & p \neq 1/3 \end{cases}$$

$\Longrightarrow$ persistent for $p=1/3$, otherwise transient.

6.3.3 a) $P = \begin{pmatrix} 1-2p & 2p & 0 \\ p & 1-2p & p \\ 0 & 2p & 1-2p \end{pmatrix}$

Calculate $P_{ij}(n)$! (& find mean reacurence time of states).

Solution:

$P = B \text{ diagonal} \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{pmatrix} B^{-1}$ $\quad P_{ij}(n) = (P^n)_{ij} = B \text{ diag} \begin{pmatrix} \lambda_1^n & & 0 \\ & \lambda_2^n & \\ 0 & & \lambda_3^n \end{pmatrix} B^{-1}$

$B = \begin{pmatrix} e_1 & e_2 & e_3 \end{pmatrix}$ where e_1, e_2, e_3 are column eigenvectors to P, and $\lambda_1, \lambda_2, \lambda_3$ are the corresponding eigenvalues.

$Pe_1 = \lambda_1 e_1 \quad Pe_2 = \lambda_2 e_2 \quad Pe_3 = \lambda_3 e_3$

where $\lambda_1, \lambda_2, \lambda_3$ solves $\det(P - \lambda I) = 0$

Solved it in Matematica (look at given paper).

6.3.4

Solution:

Four eq. and four unknown. Solve!
